

twoxtwogame Package Documentation

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1 Introduction

2×2 normal-form games are widely studied in game theory and economics. They are the simplest type of normal-form game, consisting of only two players, each with two strategies. This package provides tools for representing, visualizing, tabulating, and naming such games and their equilibrium solutions in L^AT_EX.

2 Functionality

2×2 normal-form games are parameterized with 8 variables which describe the payoffs received by each player. Therefore, the majority of functions in the `twoxtwogame` package have 8 mandatory arguments, `{a}{b}{c}...{h}`, corresponding to these variables. The first four arguments, `{a}{b}{c}{d}`, are payoffs of the row player in row-wise order, and the last four arguments, `{e}{f}{g}{h}`, are the payoffs of the column player also in row-wise order. These payoffs, G , are often represented in a tabular form, where each player has strategies A and B (Table 1a). The joint strategy distribution, σ , is parameterized with four arguments, `{a}{b}{c}{d}` (Table 1b). Some commands have additional optional keyword arguments.

2.1 Payoffs Representations

Traditionally payoffs can be represented as tables containing their raw payoff values and labeled actions. Additionally, payoffs can be represented graphically as either *best-response graphs* or *partial ordinal graphs*. These representations are compact enough to be inserted inline into text.

G	A	B
A	a, e	b, f
B	c, g	d, h

(a) Payoffs arguments.

σ	A	B
A	a	b
B	c	d

(b) Joint arguments.

Table 1: 2×2 normal-form game function arguments.

Keyword Argument	Description	Default
point offset	The distance of the points from the center	0.38em
point color	Color of points	.
point radius	Radius of points (filled circles)	0.06em
row player color	Color of row player's BR edges	.
column player color	Color of column player's BR edges	.
arrow width	Arrow head width	0.25em
arrow length	Arrow head length	0.35em
line width	Arrow line width	0.05em

Command	Output
<code>\brgraph{1}{-1}{-1}{1}{-1}{1}{1}{-1}</code>	
<code>\brgraph[point color=green]{1}{-1}{-1}{1}{-1}{1}{1}{-1}</code>	
<code>\brgraph[row player color=red, column player color=blue]{1}{-1}{-1}{1}{-1}{1}{1}{-1}</code>	

Table 2: Parameterization, default values, and example usage of the `\brgraph[...]{a}...{h}` command, which produces inline best-response graphs. Default period indicates inheritance from outer scope.

2.1.1 Best-Response Graphs

The best-response dynamics of 2×2 games are easily visualized. 2×2 games have been categorized according to these dynamics (Borm, 1987; Marris et al., 2023). Marris et al. (2023) proposed the best-response graphical representation used in this package.

The graph contains up to four directed edges: two for each player. The row player has up to two vertical directed edges which point from each of the column player's two strategies to the row player's corresponding best-response strategy (BR). Similarly, the column player has up to two horizontal edges which correspond to the best-response to each of the row player's two strategies. Self-loop edges are omitted from the graph and indicate indifference between two strategies. For example, when $a = c$. The graph can be placed inline using `\brgraph{a}...{h}`, for example: Matching Pennies , Prisoner's Dilemma , and Null $\begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}$. The graphs inherit color from its surrounding text. The graphs are customizable with optional keyword arguments (Table 2).

The graphs belong to 15 equivalence classes which have been named (Marris et al., 2023). Names for any payoff can be determined using `\brname{a}...{h}` (examples in Table 3). It is common to have the equivalent class name appear along with the best-response graph. Canonical graphs for each equivalence class are given in Table 4, and can be inserted using the commands `\<name>brgraph`. This command also accepts the same optional keyword arguments as `\brgraph[...]{a}...{h}`.

Example	Output
<code>\brname{1}{2}{3}{4}{5}{6}{7}{8}</code>	Dominant
<code>\brname{1.1}{-1}{3.7}{4}{-1.2}{3}{1}{5}</code>	Samaritan
<code>\brname{1}{-1}{-1}{1}{1}{-1}{-1}{1}</code>	Coordination
<code>\brname{-1}{1}{1}{-1}{-1}{1}{1}{-1}</code>	Coordination
<code>\brname{0}{0}{0}{0}{0}{0}{0}{1}</code>	Dress

Table 3: Example usage of the `\brname{a}...{h}` command, which produces best-response equivalence class names. Many payoffs map to the same equivalence class.

Graph Command	Graph Command	Graph Command
<code>\dominantbrgraph</code> 	<code>\safetybrgraph</code> 	<code>\heistbrgraph</code> 
<code>\coordinationbrgraph</code> 	<code>\aidosbrgraph</code> 	<code>\ignorancebrgraph</code> 
<code>\cyclebrgraph</code> 	<code>\picnicbrgraph</code> 	<code>\horseplaybrgraph</code> 
<code>\samaritanbrgraph</code> 	<code>\daredevilbrgraph</code> 	<code>\dressbrgraph</code> 
<code>\hazardbrgraph</code> 	<code>\fossickbrgraph</code> 	<code>\nullbrgraph</code> 

Table 4: Example usage of the `\<name>brgraph[...]{a}...{h}` commands, which produce the canonical graphs for the 15 equivalence classes proposed by Borm (1987) and Marris et al. (2023). These commands have the same key word arguments as `\brgraph`.

2.1.2 Ordinal Graphs and Partial Ordinal Graphs

It is also possible to visualize payoff orderings in graphical form using the `\ordgraph[<keywords>]{a}...{h}` command. Ordinal games have a simple four edge directed graph representation for each player. The “periodic table” of ordinal games is shown in Table 6. The command is also able to handle partial ordinal graphs. In the literature, these are categorized as triple, double, high, low, middle and basic ties. The representation consists of two directed graphs, one for each player. The row player’s graph is black and shifted to the bottom left, by default. The column player’s graph is gray and shifted to the top right, by default. Each graph then represents the partial ordering of the payoffs of each pure joint strategy. The ordinal representation of some common games is shown in Table 7. Examples of partially ordinal games are given in Table 8.

2.1.3 Payoff Tables

This package also provides commands for specifying payoff tables. The `\smallpayoffstable[...]{a}...{h}` command produces inline payoff tables (Table 9).

While the `\payoffstable[...]{a}...{h}` command produces full tabular payoff tables with strategy labels.

For example, `\smallpayoffstable{1}{+2}{9}{-4}{3}{2}{1}{0}` results in a small inline payoff table, $\begin{bmatrix} 1,3 & +2,2 \\ 9,1 & -4,0 \end{bmatrix}$. It is also possible to modify the values with keyword arguments (Table 9).

Larger payoff tables can be represented with the `\payoffstable[...]{a}...{h}`, `\brpayoffstable[...]{a}...{h}`, and `\ordpayoffstable[...]{a}...{h}` com-

Keyword Argument	Description	Default
point offset	The distance of the points from the center	0.38em
row player x offset	Additional horizontal offset applied to row player	-0.1e,
row player y offset	Additional vertical offset applied to row player	-0.5em
column player x offset	Additional horizontal offset applied to column player	0.1em
column player y offset	Additional vertical offset applied to column player	0.05em
point color	Color of points	.
point radius	Radius of points	0.04em
row player color	Color of row player's BR edges	.
column player color	Color of column player's BR edges	gray
arrow width	Arrow head width	0.25em
arrow length	Arrow head length	0.35em
line width	Arrow line width	0.04em

Command	Output
<code>\ordgraph{1}{-1}{-1}{1}{-1}{1}{1}{-1}</code>	
<code>\ordgraph[row player color=red, column player color=blue]{1}{-1}{-1}{1}{-1}{1}{1}{-1}</code>	

Table 5: Parameterization, default values, and example usage of the `\ordgraph[...]{a}...{h}` command, which produces inline ordinal graphs. Default period indicates inheritance from outer scope.

mands. Examples are given in Table 10.

2.2 Distribution Representations

Distributions, including joint, marginal, and conditional distributions can be visualised using either small inline graphs, tables, or larger figures.

2.2.1 Inline Distribution Graphs

Inline graphs for joint, conditional, and marginal distributions can be produced using the commands in Table 11. All the commands take a joint distribution as arguments (Table 1b). Optional arguments include `color`, `border color`, `row player color`, and `column player color`.

2.2.2 Distribution Tables

Tables can also be easily defined for joint distributions (Figure 12). Optional arguments include `label`, `color`, `border color`, `row player color`, and `column player color`.

2.2.3 Marginal Simplex

Marginals, $\sigma_r = [\sigma_r^A, \sigma_r^B]$ and $\sigma_c = [\sigma_c^A, \sigma_c^B]$, can be plotted in two dimensions and the product of two 1-simplices. Points, images, heatmaps, and quiver plots can be generated from data (Figure 1).

Chicken												
Battle												
Hero												
Compromise												
Deadlock												
Dilemma												
Hunt												
Assurance												
Coordination												
Peace												
Harmony												
Concord												
	Concord	Harmony	Peace	Coordination	Assurance	Hunt	Dilemma	Deadlock	Compromise	Hero	Battle	Chicken

Table 6: Periodic table of ordinal games (Goforth and Robinson, 2005; Robinson and Goforth, 2005) with a naming scheme proposed by Bruns (2015).

	M	F		C	D		H	T
M	3,2	0,0	C	2,2	7,0	H	+1,-1	-1,+1
F	0,0	2,3	D	0,7	4,4	T	-1,+1	+1,-1

(a) Battle of the Sexes

(b) Prisoner's Dilemma

(c) Matching Pennies

Table 7: Ordinal and partial ordinal graphs for common normal-form games.

	A	B		A	B		A	B		A	B
A	1,2	2,3	A	2,1	2,0	A	1,1	0,1	A	1,0	0,0
B	3,1	4,4	B	2,1	1,1	B	1,0	0,0	B	0,0	0,1
(a) Strict Tie			(b) Triple Tie			(c) Double Tie			(d) Basic Tie		
	A	B		A	B		A	B		A	B
A	0,1	0,1	A	3,1	3,3	A	1,0	0,-1	A	2,0	1,1
B	0,1	0,1	B	2,3	1,2	B	0,1	-1,0	B	0,2	0,0
(e) Zero Tie			(f) High Tie			(g) Middle Tie			(h) Low Tie		

Table 8: Examples of partial ordinal graphs for each of the possible ties.

2.2.4 Joint Simplex

The joint of a 2×2 normal-form game is described by four probabilities, $\{\sigma(AA), \sigma(AB), \sigma(BA), \sigma(BB)\}$. Due to the constraint that $\sum_{a \in \mathcal{A}} \sigma(a) = 1$, there is

Keyword Argument	Description	Default
row player color	Color of row player's payoffs	.
column player color	Color of column player's payoffs	.
row player first strategy color	Color of row player's first strategies	\empty
row player second strategy color	Color of row player's second strategies	\empty
column player first strategy color	Color of column player's first strategies	\empty
column player second strategy color	Color of column player's second strategies	\empty

Example	Output
<code>\smallpayoffstable{1}{2}{9}{-4}{3}{2}{1}{0}</code>	$\begin{bmatrix} 1,3 & 2,2 \\ 9,1 & -4,0 \end{bmatrix}$
<code>\smallpayoffstable[row player color=red, column player color=blue]{1}{2}{9}{-4}{3}{2}{1}{0}</code>	$\begin{bmatrix} 1,3 & 2,2 \\ 9,1 & -4,0 \end{bmatrix}$
<code>\smallpayoffstable[row player first strategy color=red, row player second strategy color=orange, column player first strategy color=purple, column player second strategy color=blue]{1}{2}{9}{-4}{3}{2}{1}{0}</code>	$\begin{bmatrix} 1,3 & 2,2 \\ 9,1 & -4,0 \end{bmatrix}$

Table 9: Parameterization, default values, and example usage of the `\smallpayoffstable[...]{a}...{h}` command, which produces inline payoff tables. Default period indicates inheritance from outer scope.

<table style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 10px;"></td><td style="padding: 2px 10px; text-align: center;">A</td><td style="padding: 2px 10px; text-align: center;">B</td></tr> <tr><td style="padding: 2px 10px; text-align: center;">A</td><td style="padding: 2px 10px; text-align: center;">1, 3</td><td style="padding: 2px 10px; text-align: center;">+2, 2</td></tr> <tr><td style="padding: 2px 10px; text-align: center;">B</td><td style="padding: 2px 10px; text-align: center;">9, 1</td><td style="padding: 2px 10px; text-align: center;">-4, 0</td></tr> </table>		A	B	A	1, 3	+2, 2	B	9, 1	-4, 0	<table style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 10px;"></td><td style="padding: 2px 10px; text-align: center;">A</td><td style="padding: 2px 10px; text-align: center;">B</td></tr> <tr><td style="padding: 2px 10px; text-align: center;">A</td><td style="padding: 2px 10px; text-align: center;">1, 3</td><td style="padding: 2px 10px; text-align: center;">+2, 2</td></tr> <tr><td style="padding: 2px 10px; text-align: center;">B</td><td style="padding: 2px 10px; text-align: center;">9, 1</td><td style="padding: 2px 10px; text-align: center;">-4, 0</td></tr> </table>		A	B	A	1, 3	+2, 2	B	9, 1	-4, 0	<table style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 10px;"></td><td style="padding: 2px 10px; text-align: center;">A</td><td style="padding: 2px 10px; text-align: center;">B</td></tr> <tr><td style="padding: 2px 10px; text-align: center;">A</td><td style="padding: 2px 10px; text-align: center;">1, 3</td><td style="padding: 2px 10px; text-align: center;">+2, 2</td></tr> <tr><td style="padding: 2px 10px; text-align: center;">B</td><td style="padding: 2px 10px; text-align: center;">9, 1</td><td style="padding: 2px 10px; text-align: center;">-4, 0</td></tr> </table>		A	B	A	1, 3	+2, 2	B	9, 1	-4, 0
	A	B																											
A	1, 3	+2, 2																											
B	9, 1	-4, 0																											
	A	B																											
A	1, 3	+2, 2																											
B	9, 1	-4, 0																											
	A	B																											
A	1, 3	+2, 2																											
B	9, 1	-4, 0																											
(a) <code>\payoffstable</code>	(b) <code>\brpayoffstable</code>	(c) <code>\ordpayoffstable</code>																											

Table 10: Examples of the three table commands.

Command Signature	Example Command	
<code>\jointgraph[...]{a}...{d}</code>	<code>\jointgraph{.4}{.3}{.1}{.2}</code>	
<code>\rowcondgraph[...]{a}...{d}</code>	<code>\rowcondgraph{.4}{.3}{.1}{.2}</code>	
<code>\colcondgraph[...]{a}...{d}</code>	<code>\colcondgraph{.4}{.3}{.1}{.2}</code>	
<code>\marginalgraph[...]{a}...{d}</code>	<code>\marginalgraph{.4}{.3}{.1}{.2}</code>	
<code>\jmgraph[...]{a}...{d}</code>	<code>\jmgraph{.4}{.3}{.1}{.2}</code>	

Table 11: Inline distribution graphs.

redundancy in this parameterization. Therefore probability distributions (a 4-dimensional barycentric coordinate system) can be visualized on a 3-simplex in 3-dimensional Cartesian coordinates.

A joint distribution is a point within this simplex, with pure joint strategies corresponding to the vertices of the simplex. Sets of points can also be visualized. In particular, convex polytope sets can be visualized by tracing the edges of the polytope. Games permit a convex polytope of (C)CE equilibria, and therefore these can be easily visualized. In fact, in two-strategy games,

<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px;">A</td><td style="padding: 2px;">0.1</td><td style="padding: 2px;">0.2</td></tr> <tr><td style="padding: 2px;">B</td><td style="padding: 2px;">0.3</td><td style="padding: 2px;">0.4</td></tr> </table>	A	0.1	0.2	B	0.3	0.4	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px;">A</td><td style="padding: 2px;">0.1</td><td style="padding: 2px;">0.2</td></tr> <tr><td style="padding: 2px;">B</td><td style="padding: 2px;">0.3</td><td style="padding: 2px;">0.4</td></tr> </table>	A	0.1	0.2	B	0.3	0.4	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px;">A</td><td style="padding: 2px;">0.1</td><td style="padding: 2px;">0.2</td></tr> <tr><td style="padding: 2px;">B</td><td style="padding: 2px;">0.3</td><td style="padding: 2px;">0.4</td></tr> </table>	A	0.1	0.2	B	0.3	0.4
A	0.1	0.2																		
B	0.3	0.4																		
A	0.1	0.2																		
B	0.3	0.4																		
A	0.1	0.2																		
B	0.3	0.4																		
(a) <code>\jgjointtable</code>	(b) <code>\rcgjointtable</code>	(c) <code>\ccgjointtable</code>																		
<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px;">A</td><td style="padding: 2px;">0.1</td><td style="padding: 2px;">0.2</td></tr> <tr><td style="padding: 2px;">B</td><td style="padding: 2px;">0.3</td><td style="padding: 2px;">0.4</td></tr> </table>	A	0.1	0.2	B	0.3	0.4	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px;">A</td><td style="padding: 2px;">0.1</td><td style="padding: 2px;">0.2</td></tr> <tr><td style="padding: 2px;">B</td><td style="padding: 2px;">0.3</td><td style="padding: 2px;">0.4</td></tr> </table>	A	0.1	0.2	B	0.3	0.4	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px;">A</td><td style="padding: 2px;">0.1</td><td style="padding: 2px;">0.2</td></tr> <tr><td style="padding: 2px;">B</td><td style="padding: 2px;">0.3</td><td style="padding: 2px;">0.4</td></tr> </table>	A	0.1	0.2	B	0.3	0.4
A	0.1	0.2																		
B	0.3	0.4																		
A	0.1	0.2																		
B	0.3	0.4																		
A	0.1	0.2																		
B	0.3	0.4																		
(d) <code>\mgjointtable</code>	(e) <code>\jmgjointtable</code>	(f) <code>\jointtable</code>																		

Table 12: Examples of joint distribution table commands. The naming convention is: “jg” for “joint graph”, “rcg” for “row conditional graph”, “ccg” for “column conditional graph”, “mg” for “marginal graph”, and “jmg” for “joint-marginal graph”.

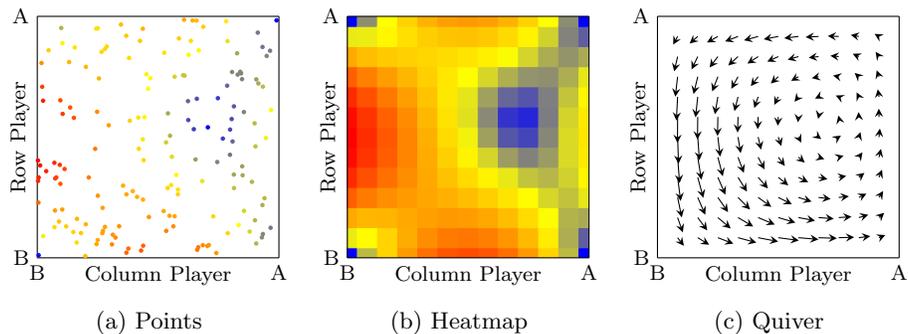


Figure 1: Marginal Simplex Plots

CEs and CCEs are equivalent. NEs are equilibria that are (C)CEs and also factorize, $\sigma(a_1, a_2) = \sigma_1(a_1)\sigma_2(a_2)$. In 2×2 normal-form games, NEs can be either points, lines, or in a degenerate case a manifold which stretches across all the vertices of the simplex. These polytopes can be visualized using the `polytope[...]{a}...{h}` command.

2.3 Embeddings

The equilibrium-invariant embeddings introduced by Marris et al. (2023) can be visualized using the `nontrivialemb[...]` command.

3 Tips and Tricks

Slow Builds or Out of Memory Errors Some of the figures created by this package can take time to compile, and may slow down compile times for your project. Consider using the externalize command of TikZ/PFGplots to

Keyword Argument	Description	Default
width	Width of plot	<code>\linewidth</code>
image	File path to background image	empty
matrix	File path to background matrix data	empty
points	File path to background points data	empty
axis color		black
partial trivial		false
symmetric strategies		false
symmetric players		false
no class boundaries	Suppress class boundaries	false
no equilibrium boundaries	Suppress equilibrium boundaries	false
no best-response graphs	Suppress false best-response graphs	false
best-response graph color		black
best-response graph font size		<code>\scriptsize</code>
no best-response names	Suppress best-response names	false
best-response name orientation		-90
best-response name distance		-0.1em
best-response name color		black!70
best-response name font size		<code>\tiny</code>
no axes labels	Suppress axes labels	false
no tick labels	Suppress tick labels	false

Table 13: `embedding[...]` command parameterization.

mitigate this issue:

```
\usepackage{tikz}
\usetikzlibrary{external}
\tikzexternalize[prefix=cache/, up to date check=md5]
```

This will sequentially build each each TikZ picture individually and cache it. Subsequent project builds will use the cache rather than rebuild the picture each time. When modifying commands, it may be necessary to clear the cache to regenerate the pictures. Some setups may require manually making the cache directory and adding a dummy file in that directory. This process may be slower on the first build run. If your build times out trying again may work (because it will reuse the pictures that were cached on the first run).

4 Acknowledgements

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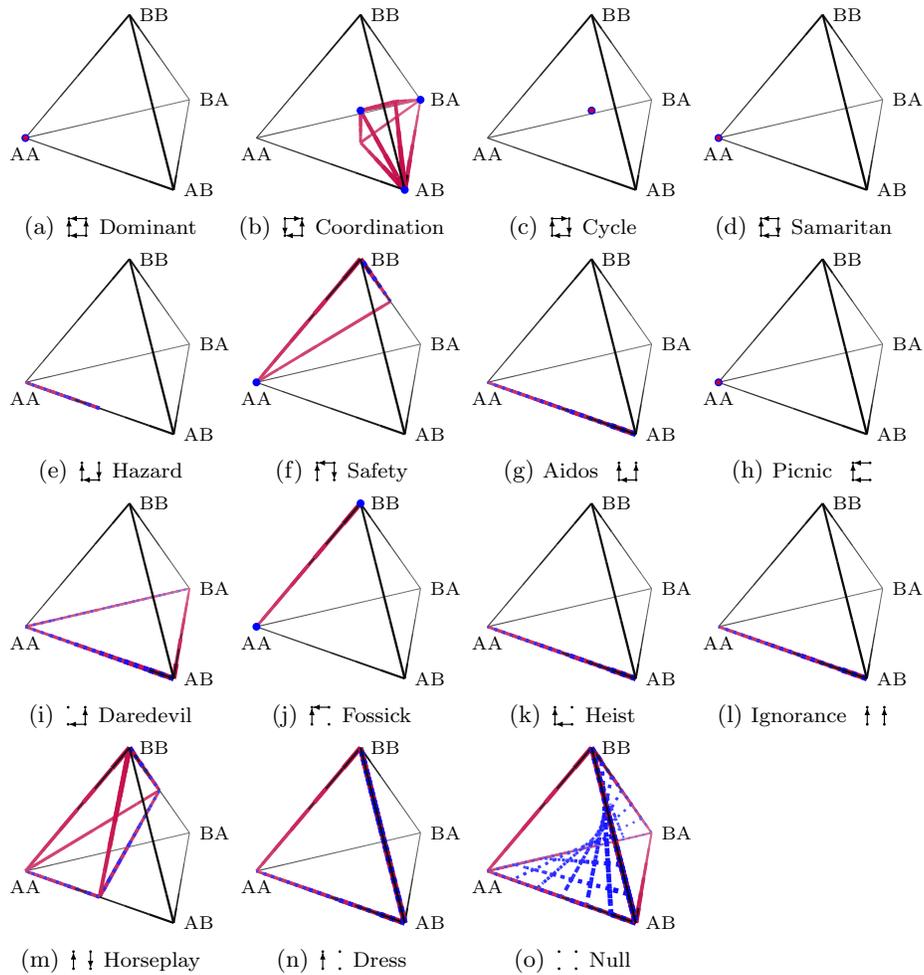


Figure 2: Equilibrium polytopes of the canonical games introduced by Marris et al. (2023). The black lines represent edges of the joint probability simplex. The solid purple lines represent the edges of the (C)CE polytope. The dashed blue lines represent a continuum of NE. Points with respective colours represent (C)CE and NE point solutions.

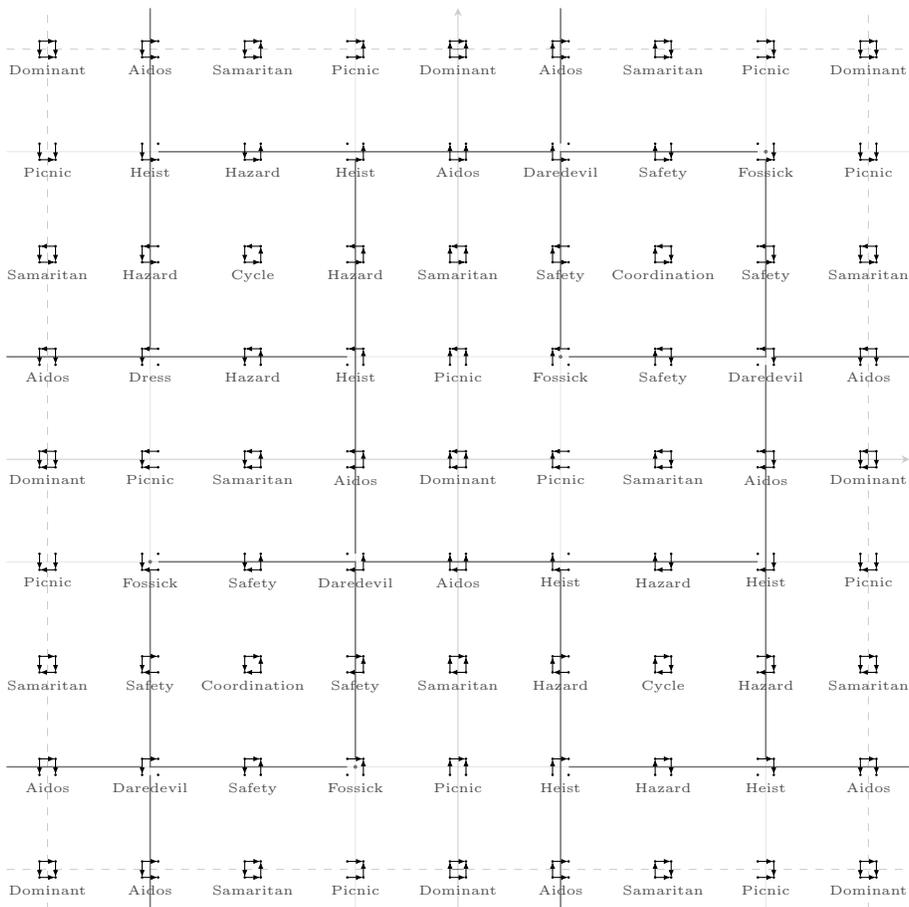
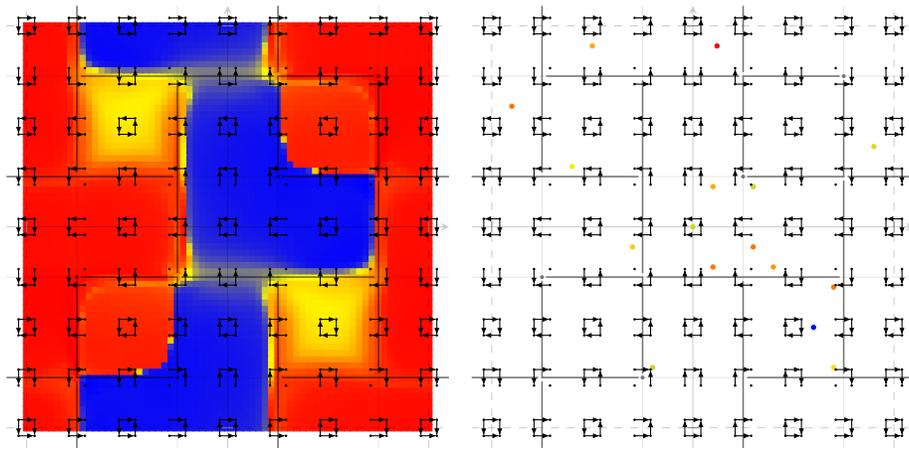


Figure 3: Embeddings plot.



(a) Heatmap. Data from Liu et al. (2024).

(b) Points

Figure 4: Embedding plots with different backgrounds.

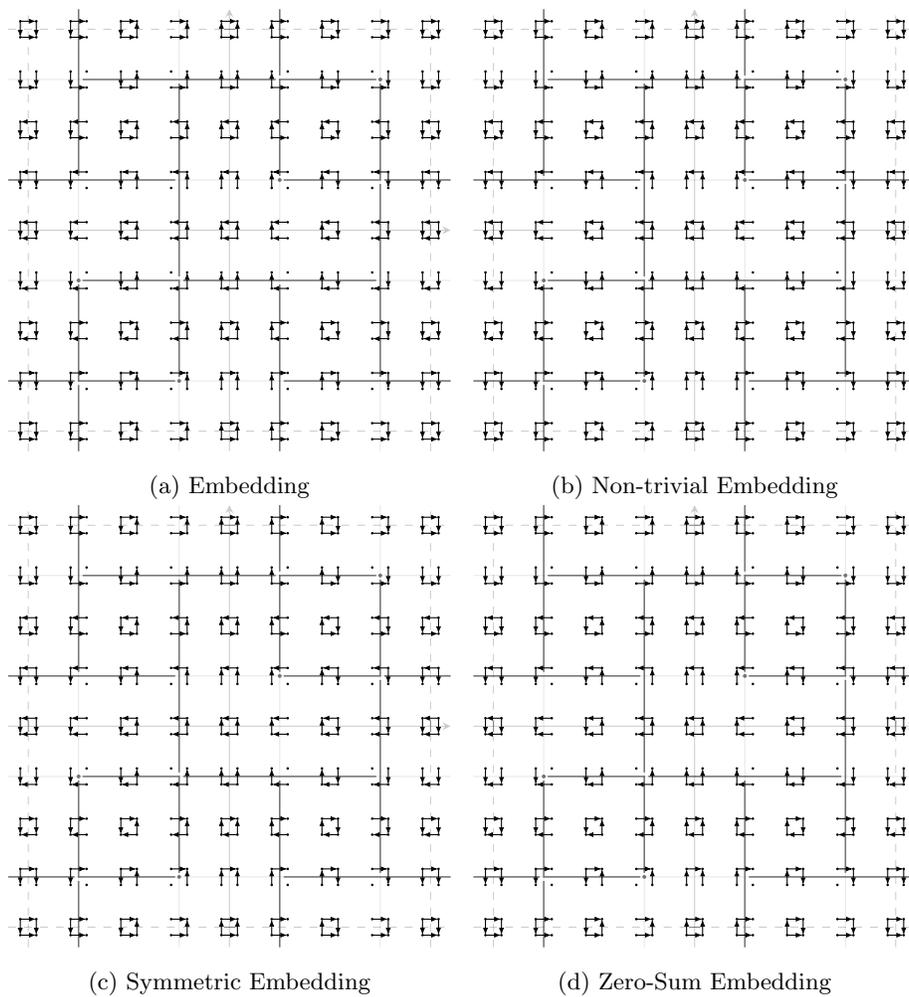


Figure 5: Embedding Visualizations.